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Estimation of systematic error in an equatorial ocean model using data assimilation

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SUMMARY

Assimilation of temperature observations into an ocean model near the equator often results in a dynamically unbalanced state with unrealistic overturning circulations. The way in which these circulations arise from systematic errors in the model or its forcing is discussed. A scheme is proposed, based on the theory of state augmentation, which uses the departures of the model state from the observations to update slowly evolving bias fields. Results are summarized from an experiment applying this bias correction scheme to an ocean general circulation model. They show that the method produces more balanced analyses and a better fit to the temperature observations. © Crown copyright 2002. Reproduced with the permission of Her Majesty's Stationery Office. Published by John Wiley & Sons, Ltd.

KEY WORDS: data assimilation; ocean model; systematic model error

1. INTRODUCTION

Assimilation of data into ocean models is becoming increasingly viable. Substantial *in situ* real-time observing networks such as the TAO buoy array [1], and the Argo (1999) array of autonomous profiling floats have been or are being deployed. Satellite measurements of the surface wind stress and sea surface height are also available for assimilation by operational centres.

A particular problem for seasonal forecasting is that, unless particular care is taken, ocean models do not retain the observational data assimilated into them for more than a few months but drift away towards their own climatologies in the key regions of variability within a few degrees of the equator. There have been a number of studies to address problems of systematic model errors. Of particular relevance to this paper, are some studies in the context of Kalman filters [2]. Friedland [3], proposed augmenting the state vector by a model bias vector and transforming the gain matrices of the Kalman filter to produce a computationally cheaper method. Dee and Da Silva [4], hereafter referred to as DDS, have applied this idea to numerical weather prediction (NWP). In this paper we build on the ideas of DDS and Griffith

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and Nichols [5], by developing a formulation of the augmented state method to produce balanced analyses near the equator and show results of the application of the method.

2. PROBLEMS ARISING FROM DATA ASSIMILATION NEAR THE EQUATOR

In normal years easterlies blow along the equator in the Pacific. The surface wind stress is mixed down only over the top 50–100 m of the ocean and to oppose this stress by a suitable near-surface pressure gradient the surface of the ocean tilts so that it is higher in the west than the east. This pressure gradient reduces markedly with depth because the water on the eastern side of the Pacific in the upper ocean is colder than that on the western side. In El Niño years the easterly wind stresses are weaker and the surface pressure gradients and sub-surface density gradients weaken to maintain the overall balance.

The horizontal momentum equations can be written as

$$\rho\left(\frac{\partial u}{\partial t}+\Gamma(u)-fv\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xz}}{\partial z} \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t}+\Gamma(v)+fu\right)=-\frac{\partial p}{\partial y}+\frac{\partial \tau_{yz}}{\partial z} \tag{2}$$

where (x, y) and (u, v) are the eastward and northward co-ordinates and velocities, respectively, (τ_x, τ_y) are the corresponding components of the wind forcing and Γ represents an advective operator. In the equatorial region, the terms on the left-hand side of these equations are small compared to the pressure gradients and wind stress gradients so that

$$\frac{\partial p}{\partial x}\approx\frac{\partial \tau_{xz}}{\partial z},\quad \frac{\partial p}{\partial y}\approx\frac{\partial \tau_{yz}}{\partial z} \tag{3}$$

The main dynamical balance along the equator is therefore between the wind stress and the pressure gradients. Any disruption to this balance will induce unrealistic horizontal velocities which, through continuity, will result in large vertical velocities.

Data assimilation disrupts the balance between pressure gradients and wind stress in the case when the wind stress driving the ocean model is too weak. In this case, assimilation increments to the density field over-strengthen the subsurface pressure gradient. Similarly unbalanced pressure gradients could result if the parameterization of the downward vertical mixing of momentum input by the wind stress were flawed. Equatorial ocean models were enormously improved by Pacanowski and Philander's parameterization of this process [6], but as the parameterization is very difficult it is likely to have significant residual errors.

The problems which arise near the equator are illustrated using two integrations described in detail in Reference [7] with the Forecasting Ocean–Atmosphere Model (FOAM). The first assimilates surface temperature and thermal profile data. The second is identical to the first except that no data is assimilated. The integrations start from a state of rest with potential temperatures and salinities derived from the Levitus (1994) climatology [8], for the 1st May. They are forced by monthly mean climatological fluxes. The wind stresses are taken from Hellerman and Rosenstein [9], which are generally regarded as being too strong in the equatorial regions. The data assimilation component is based on the analysis correction scheme

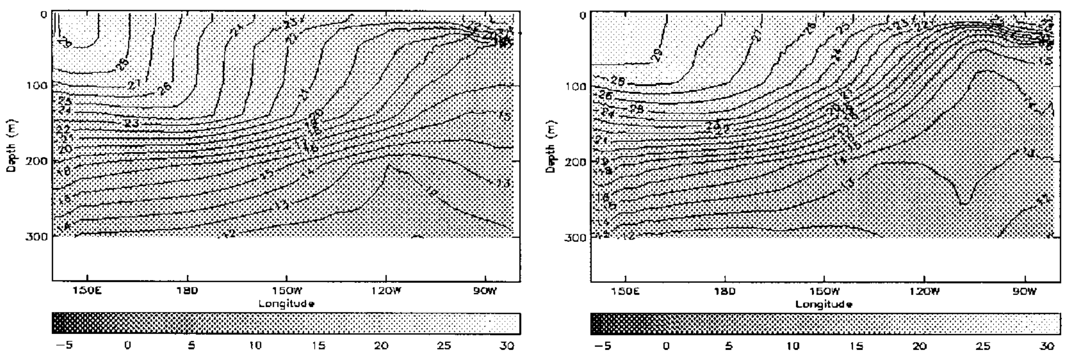

Figure 1. Annual mean (second year) potential temperature (°C) cross-section along equator between 140°E and 90°W: (a) control without data assimilation and (b) with data assimilation.

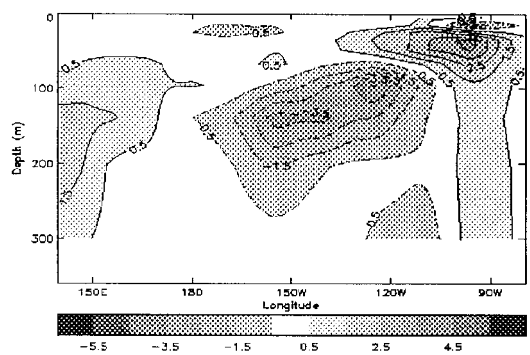

Figure 2. Annual mean (second year) potential temperature increments (°C month^{-1}) cross-section along equator between 140°E and 90°W for run with data assimilation.

of Lorenc *et al.* [10]. No salinity data are used and no salinity increments are made by the assimilation scheme. Observations valid from the 1st May 1995 to 30th April 1996 were assimilated. Neither an El Niño nor a La Niña event occurred during this period. The integrations presented ran for two years. The data assimilation run assimilated the same data in both years of integration.

Figures 1(a) and 1(b) show the time mean of the potential temperature field along the equator in the Pacific for the second year of the control and assimilation integrations, respectively. It is clear from these plots that the assimilation acts to tighten the thermocline so that the temperature gradient is much steeper with data assimilation, as required.

Figure 2 shows the time-mean cross-section of the potential temperature increments made by the data assimilation component during the second year of the assimilation along the same section as Figure 1. The increments are very large, exceeding 3°C month^{-1} over large areas. This indicates that in these regions over the course of the second year of integration the data assimilation warmed/cooled the ocean by more than 36°C and the ocean model cooled/warmed the ocean by similar amounts. The density change caused by the input of heat

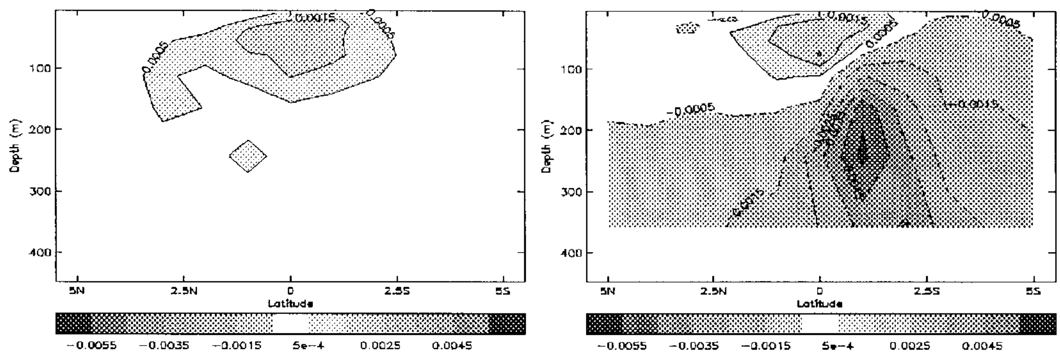

Figure 3. Annual mean (second year) vertical velocities (cm s^{-1}) cross-section across equator at 110°W: (a) control without data assimilation and (b) with data assimilation.

by the assimilation induces persistent vertical advection in the model. An example of this is shown in Figure 3. Thus, the ocean model cannot be considered to be in a satisfactory dynamical balance during the assimilation, and any forecast made from an assimilation state is likely to drift very rapidly during the first months of the forecast.

In the horizontal velocity fields (not shown here), the wind-driven near surface westward current and the eastward equatorial undercurrent are present in the control integration. In the assimilation experiment, the undercurrent does not penetrate as far to the east in the model as it should do and the surface currents are reversed in part of the region.

3. FORMULATION OF METHOD

3.1. Augmented state treatment of models with systematic errors

For models containing systematic errors, a standard approach is to augment the model state with a set of systematic model error variables [3] and DDS. This section presents the main ideas involved in this approach.

The evolution of the true state of the ocean, $\mathbf{x}_k^t \in \mathbb{R}^n$, from time t_k to t_{k+1} is taken to be described by the stochastic vector difference equation

$$\mathbf{x}_{k+1}^t = M^t(\mathbf{x}_k^t, \mathbf{u}_k^t) + \eta_k^t \tag{4}$$

where superscript t represents true fields, $M^t: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the true system operator, $\mathbf{u}_k^t \in \mathbb{R}^m$ is the vector of true model inputs (such as surface wind stress) and $\eta_k^t \in \mathbb{R}^n$ is a vector of random disturbances, assumed to form a white Gaussian sequence.

The observations are assumed to be given by the equation

$$\mathbf{y}_k = H_k(\mathbf{x}_k^t) + \varepsilon_k \tag{5}$$

where $\mathbf{y}_k \in \mathbb{R}^{p_k}$ is the vector of observations available at time t_k which is related to the true state of the system through the observation operator $H_k : \mathbb{R}^n \rightarrow \mathbb{R}^{p_k}$ and contains random errors $\varepsilon_k \in \mathbb{R}^{p_k}$ which are assumed to form a white Gaussian sequence.

The equation which models the system can be written as

$$\mathbf{x}_{k+1} = M^m(\mathbf{x}_k, \mathbf{u}_k) + \eta_k^m \tag{6}$$

where superscript m represents model fields, $M^m: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the model system operator and $\eta_k^m \in \mathbb{R}^n$ is a vector of random disturbances, assumed to form a white Gaussian sequence.

It is usually assumed that the forecast model is deterministic, $\eta_k^m \equiv 0$; is perfect, $M^m \equiv M^t$; and has perfect inputs, $\mathbf{u}_k \equiv \mathbf{u}_k^t$. In this case, the normal data assimilation process may be applied and the analysed solution can be made to converge to the true solution over time by suitable choices of the gain matrix, given certain conditions [11]. For ocean models, these assumptions about the forecast model are not valid. This is also true for many other applications such as humidity fields in NWP. When modelling systems such as the oceans, the exact representation of the true system on the model grid will not be known, and approximations to the true operator have to be made. Also, the inputs to the model are not always known accurately.

We now suppose instead that the model used to propagate the state variables and inputs contains systematic errors. We write this assumption as

$$M^m(\mathbf{x}_k, \mathbf{u}_k) = M^t(\mathbf{x}_k, \mathbf{u}_k) + T(\mathbf{b}_k) \tag{7}$$

where $\mathbf{b}_k \in \mathbb{R}^q$ and $T: \mathbb{R}^q \to \mathbb{R}^n$ is some operator which is to be chosen. The vector $\mathbf{b}_k$ is not strictly the systematic model error vector after introducing the operator T. However, we call this vector the model bias in this section for convenience. The operator is included because it is possible that only certain parts of the model will contain systematic errors, i.e. $q < n$. T is usually taken to be the identity as in DDS but there are other possible choices. In the pressure correction method described in the following subsection, we take the operator T to be of a specific form, based on our understanding of the nature of the systematic errors.

We assume that the evolution of the model biases is governed by the stochastic vector difference equation

$$\mathbf{b}_{k+1} = W(\mathbf{b}_k, \mathbf{x}_k) + \zeta_k \tag{8}$$

where $W: \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^q$ evolves the model bias variables and $\zeta_k \in \mathbb{R}^q$ forms a white Gaussian sequence.

If the normal data assimilation process is applied to the system with systematic model errors, the analysed solution will not converge to the true solution as time increases. If we augment the state vector with the vector of model bias variables and apply the data assimilation process to this augmented state however, then it can be shown in the linear case that the analysed state vector can be made to converge to the true state vector, given similar conditions to those for the normal data assimilation problem [12]. This convergence assumes that we know how the systematic model error evolves in time.

The idea of state augmentation can be applied to any of the data assimilation methods. In the case of Optimal Interpolation [13], the analysis step can be written as

$$\mathbf{x}_k^{\mathrm{a}} = \mathbf{x}_k^{\mathrm{f}} + K_k^x[\mathbf{y}_k - H_k(\mathbf{x}_k^{\mathrm{f}})] \tag{9}$$

$$\mathbf{b}_k^{\mathrm{a}} = \mathbf{b}_k^{\mathrm{f}} + K_k^{\mathrm{b}}[\mathbf{y}_k - H_k(\mathbf{x}_k^{\mathrm{f}})] \tag{10}$$

where the gain matrices are determined by

$$K_k^x = P_{(xx)}^{\mathrm{f}}(t_k)H_k^T[H_k P_{(xx)}^{\mathrm{f}}(t_k)H_k^T + R_k]^{-1} \tag{11}$$

$$K_k^{\mathrm{b}} = P_{(bx)}^{\mathrm{f}}(t_k)H_k^T[H_k P_{(xx)}^{\mathrm{f}}(t_k)H_k^T + R_k]^{-1} \tag{12}$$

Here, $P_{(xx)}^{\mathrm{f}}(t_k) \in \mathbb{R}^{n\times n}$ is the forecast error covariance matrix for the state variables, $P_{(bx)}^{\mathrm{f}}(t_k) \in \mathbb{R}^{q\times n}$ is the cross-covariance matrix between the errors in the state and errors in the model bias variables, $H_k \in \mathbb{R}^{p_k \times n}$ is the linearized observations operator and $R_k \in \mathbb{R}^{p_k \times p_k}$ is the observation error covariance matrix.

As well as converging to the true state vector, a potential advantage of this method is that we obtain an estimate of the errors in the model which might help to improve its weaker components. A difficulty of the method is that we may not know exactly how the systematic model errors evolve. If we can make a reasonable estimate, however, the analysis of the augmented state should provide a better analysis than that without the systematic model error correction.

3.2. The pressure correction method

We now introduce specific choices for K_k^{b}, T and W for the bias correction, aimed at reducing the effects of the systematic errors described in Section 2, that is to restore the balance between the surface wind stress and the model's pressure gradients. We assume that we have observations of potential temperature and salinity only, and use these observations to produce bias fields which will correct the model's pressure gradients.

In the normal data assimilation procedure, increments to the potential temperature and salinity fields are made using the differences between the observed and model forecast fields. To produce estimates of the bias in these model fields, an analysis of the form of Equation (10) is performed, i.e.

$$\theta_k^{\mathrm{ba}} = \theta_k^{\mathrm{bf}} + K_k^{\mathrm{b}\theta}[\theta_k^{\mathrm{o}} - H_k^{\theta}(\theta_k^{\mathrm{f}})] \tag{13}$$

$$\mathbf{S}_k^{\mathrm{ba}} = \mathbf{S}_k^{\mathrm{bf}} + K_k^{\mathrm{bS}}[\mathbf{S}_k^{\mathrm{o}} - H_k^{S}(\mathbf{S}_k^{\mathrm{f}})] \tag{14}$$

where θ_k and $\mathbf{S}_k$ are the potential temperature and salinity, respectively, at time t_k, superscript b indicates a bias field, superscript a indicates an analysis, superscript f indicates a forecast, superscript o denotes observations, H_k^{θ} interpolates from the model grid to the temperature observations' positions and H_k^S interpolates from the model grid to the salinity observations' positions. The forecast model of θ_k^{b} and $\mathbf{S}_k^{\mathrm{b}}$ is assumed in this paper to be constant, so that

$$\theta_{k+1}^{\mathrm{bf}} = \theta_k^{\mathrm{ba}} \tag{15}$$

$$\mathbf{S}_{k+1}^{\mathrm{bf}} = \mathbf{S}_k^{\mathrm{ba}} \tag{16}$$

although this can be altered if something is known about how the bias fields evolve. Initially, the temperature and salinity bias fields are set to zero, i.e. $\theta_0^{\mathrm{ba}} = \mathbf{S}_0^{\mathrm{ba}} = 0$.

The expression for the gain matrix for the bias variables written in Equation (12) involves the cross-covariance matrix for the errors in the state and model bias variables. Statistics for this calculation would be very difficult to obtain so we choose this matrix to be of a simple form,

$$K_k^{\mathrm{b}} = -\alpha K_k^x \tag{17}$$

where α is a constant between zero and one.

The estimates of the biases in the temperature and salinity fields described in Equations (13) and (14) should now be used to calculate a bias in the pressure field so that the balance described in Section 2 between the pressure gradient and wind forcing can be restored. This is done by making use of the operator T in Equation (7), so that the bias fields are not simply added onto the model equations. Instead, a biased density ρ_k^{b} is calculated through the equation of state,

$$\rho_k^{\mathrm{b}} = \rho(\theta_k^{ma} + \theta_k^{\mathrm{ba}}, \mathbf{S}_k^{ma} + \mathbf{S}_k^{\mathrm{ba}}) - \rho_k^m \tag{18}$$

and this is used through the hydrostatic equation to produce a compensating model 'pressure' field,

$$p_k^{\mathrm{b}}(z) = \int_{-H}^{z} -\rho_k^{\mathrm{b}} g \, \mathrm{d}z \tag{19}$$

where z is the depth, H is the depth of the ocean and g is the gravitational constant. The pressure at the bottom of the model is kept the same during this calculation. Also, the model's potential temperature and salinity fields are not altered during this operation.

The corrected pressure field is now used in the horizontal momentum equations which can be written in the continuous case as

$$\rho(\partial u^m/\partial t + \Gamma(u^m) - f v^m) = -\partial(p^m + p^{\mathrm{b}})/\partial x + \partial \tau_{xz}/\partial z \tag{20}$$

$$\rho(\partial v^m/\partial t + \Gamma(v^m) + f u^m) = -\partial(p^m + p^{\mathrm{b}})/\partial y + \partial \tau_{yz}/\partial z \tag{21}$$

so that the compensating pressure field should restore the balance between the pressure gradients and the wind stress described in Section 2.

4. RESULTS

An analysis of the pressure correction method applied to the linear shallow water equations on a β-plane is given in Reference [14]. This shows that with complete observational coverage of the surface height field, the pressure correction method will ensure that the density, pressure and vertical velocity fields will all match those of the true solution even when the solution is driven by incorrect wind stresses. In the absence of pressure correction, none of these fields match the true solution.

The integration assimilating data described in Section 2 has been repeated using the pressure correction scheme as described in Section 3.2. The weighting of the error covariance described in Equation (17) is chosen to be (i) $\alpha = 0.1$ and (ii) $\alpha = 0.3$. Figures 4(a) and 4(b) show the

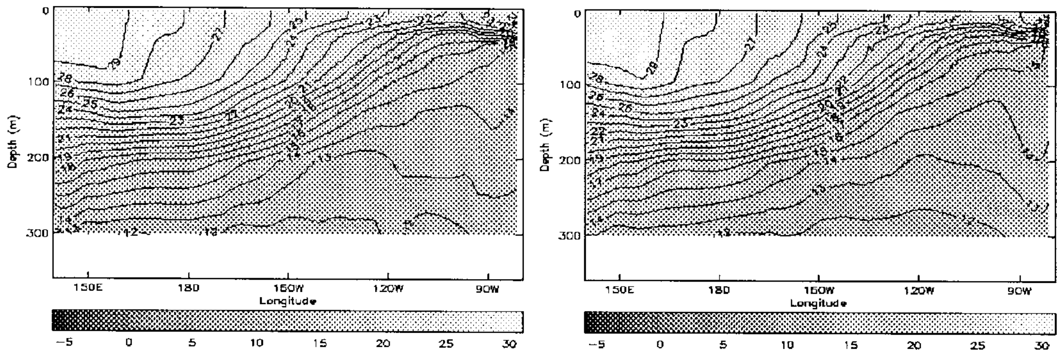

Figure 4. Annual mean (second year) potential temperature (°C) cross-section along equator between 140°E and 90°W: (a) pressure correction, $\alpha = 0.1$ and (b) pressure correction, $\alpha = 0.3$.

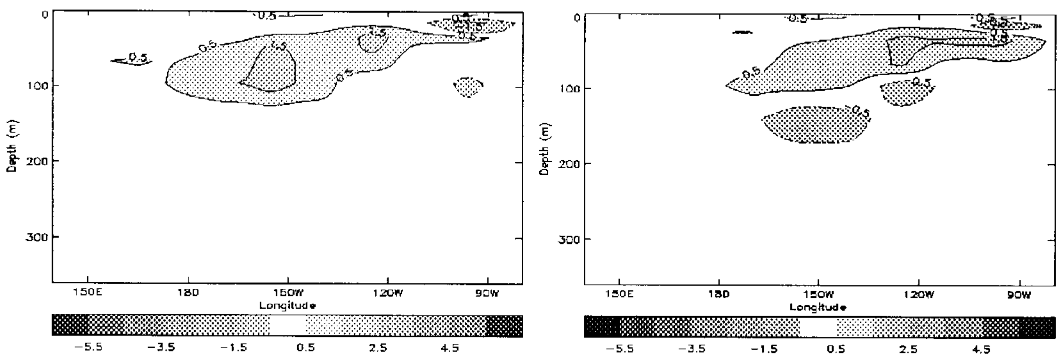

Figure 5. Annual mean (second year) potential temperature increments (°C month^{-1}) cross-section along equator between 140°E and 90°W: (a) pressure correction, $\alpha = 0.1$ and (b) pressure correction, $\alpha = 0.3$.

time mean potential temperature fields for these two integrations in a form which enables direct comparison with the results discussed in Section 2. Figures 1(b) and 4(a) and 4(b) are generally in quite close agreement—the thermocline still contains the tight temperature gradients—but there are discernible differences particularly below 100 m depth near 90°W.

The time mean potential temperature increment fields from the two pressure correction runs along the equator in the Pacific are shown in Figure 5. It is clear that the temperature increments are much smaller in magnitude for the runs with the pressure correction included than for the standard integration shown in Figure 2, and are constrained to the top 150 m. The maximum values for the pressure corrected runs are about 2°C per month compared with 4°C per month for the run with normal data assimilation.

Figure 6 shows that the large annual mean vertical velocities below 150 m depth which were present in the run with normal data assimilation at the 110°W cross-section have been eliminated. For the horizontal velocities (not shown here), the maximum speed in the equatorial undercurrent and increased eastward penetration of the current in both pressure correction

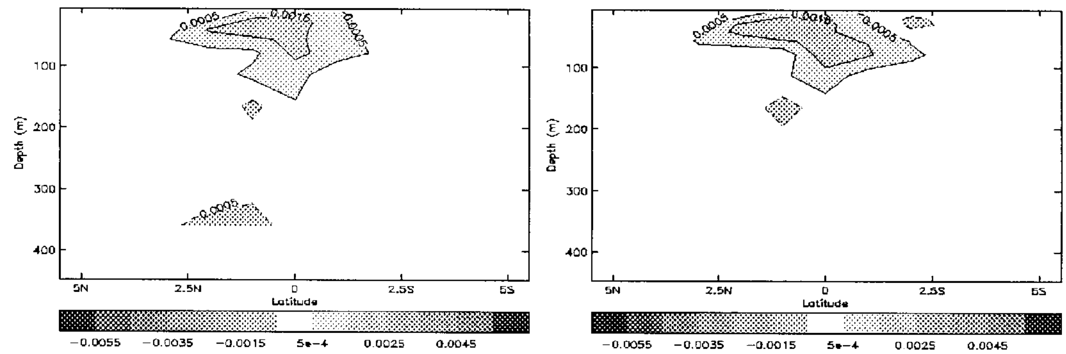

Figure 6. Annual mean (second year) vertical velocities (cm s^{-1}) cross-section across equator at 110°W: (a) pressure correction, $\alpha = 0.1$ and (b) pressure correction, $\alpha = 0.3$.

integrations are significant improvements over both the control and standard assimilation integrations. The surface currents are now of similar structure to the control, with the reversal of the surface currents in the run with normal data assimilation having been eliminated.

5. CONCLUDING SUMMARY

It has been shown that simple assimilation of thermal profile data from the equatorial Pacific into an OGCM can result in unrealistically strong vertical motions near the equator which counteract some of the increments made by the data assimilation scheme and lead to similar increments to the model state being repeatedly made by the assimilation scheme. This problem is explained to be due to difficulties in representing the momentum balance in the equatorial oceans which can lead to an ocean state with a biased density field and hence unbalanced density increments.

The pressure correction method is presented as an application of the general bias correction theory which, motivated by the dynamics, makes specific choices for estimates of the covariances between errors in the model state and the model bias. In particular, it assumes that the biases in the model equations are confined to the momentum equations and amends them only by addition of a pressure gradient bias field. The results of experiments with the FOAM system show that, when using the pressure correction method, the time mean vertical velocities and temperature increments are smaller than the standard assimilation scheme, indicating that the pressure correction produces a more balanced ocean state.

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